Noise Sensitivity of Sparse Signal Representations

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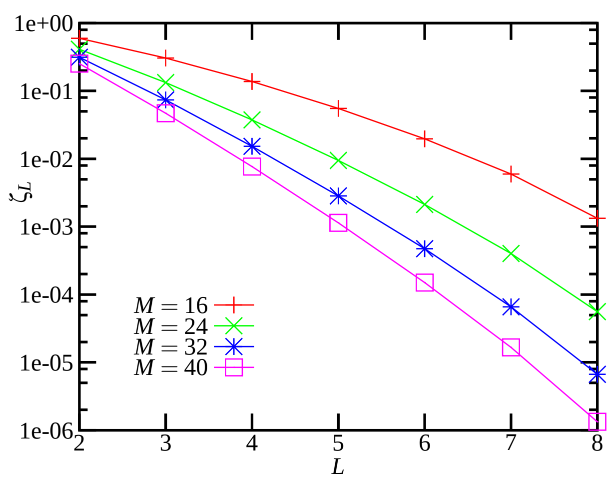
Fourier and wavelet transforms, ubiquitous in signal processing and data analysis, represent a signal as a linear combination of vectors from a basis set which is *complete*, meaning that the number of basis vectors is such that there is no redundancy, with exactly one possible representation for every signal to be decomposed. An alternative approach, which has recently been the subject of considerable research, is to decompose a signal onto an *overcomplete* set, usually referred to as the *dictionary*, in which the number of basis vectors (or *atoms*) is such that there are many different possible representations for a specific signal. To make the decomposition well-defined, it is necessary to define some criterion for selecting one particular solution. The simplest of these is the minimum l^2 norm solution, which corresponds to the pseudo-inverse of the matrix with the atoms of the dictionary as its columns. Minimum sparsity, the number of non-zero coefficients in the solution, offers significant advantages as an optimality criterion. Such sparse representations have found a number of applications [1], including EEG (electroencephalography) and MEG (magnetoencephalography) estimation [2], time-frequency analysis [3], and spectrum estimation [4].

Sparse representations are of particular interest when one has reason, based on physics or other prior knowledge, to expect the signals in question to consist of a superposition of only a few fundamental functions, the coefficients of which are significant. In this case, it is useful to know when the sparse decomposition of the signal may be expected to correspond to the original generating coefficients. A number of recent uniqueness results [2, 5, 6, 7, 8] provide conditions under which a unique sparse decomposition exists, so that exact reconstruction of the original generating coefficients is possible in the absence of any noise.

These results, however, are of little assistance when the signal is known to include a noise component. Under these more realistic conditions, one would like to bound the reconstruction error in terms of the signal noise magnitude (that is, given a bound on the size of the noise in the signal, provide a bound on the maximum distance between any two appropriate sparse reprentations of that signal). The construction of such bounds is described in detail in a recent publication [9]. Of these bounds, the simplest to describe is constructed in terms of ζ_L, a measure of the stability of the linear independence of L-sized subsets of atoms of a dictionary; given signal $\mathbf{s}$ with sparse representation α and signal $\mathbf{s}'$ with sparse representation β, one has the bound

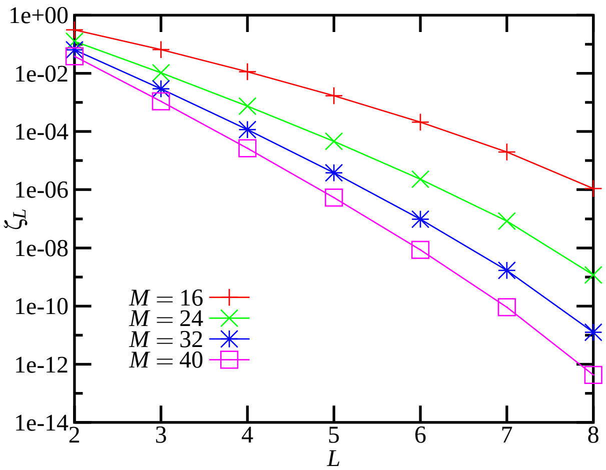
$$\|\Delta\alpha\| \leq \zeta_L^{-1}\|\Delta\mathbf{s}\|$$

on the difference $\Delta\alpha$ between the two solutions in terms of the difference $\Delta\mathbf{s}$ between the two signals, where L is the sum of the number of non-zero coefficients in solutions α and β.



Graph illustrating the increase in noise sensitivity with decreasing sparsity for DFT dictionaries with M atoms, each of which is a vector of 8 coefficients. The sparse representation error magnitude may be as large as ζ_L^{-1} times the signal error magnitude.

The figures compare ζ_L values for two example overcomplete dictionaries; one based on the Discrete Fourier Transform (DFT) and the other



Graph illustrating the increase in noise sensitivity with decreasing sparsity for DCT dictionaries with M atoms, each of which is a vector of 8 coefficients. The sparse representation error magnitude may be as large as ζ_L^{-1} *times the signal error magnitude.*

on the Discrete Cosine Transform (DCT). Except at very low noise levels, very high degrees of sparsity, or small overcompleteness factors, these results indicate very high noise sensitivities for the common DFT and DCT dictionaries. Reconstructions with respect to the overcomplete DCT dictionary are vastly more noise sensitive than those with respect to the overcomplete DFT dictionary. In superresolution applications using overcomplete sinusoidal dictionaries [4], for example, these results allow an explicit quantification of the tradeoff between spectral resolution (depending on the degree of overcompleteness of the dictionary) and noise sensitivity of the result, and also suggest that the DFT dictionary is a better choice for superresolution than the DCT dictionary due to the significantly lower noise sensitivity of the former dictionary.

Acknowledgements

Los Alamos Report LA-UR-02-1827. Funded by the Department of Energy under contracts W-7405-ENG-36 and the DOE Office of Science Advanced Computing Research (ASCR) program in Applied Mathematical Sciences.

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